PERIOD-IV

M12CH15

CHAPTER

15 Algebraic Process

15.1. SIMPLIFICATION

Simplification is a process of replacing a mathematical expression by an equivalent one, that is simpler (usually shorter) for example. Simplification of algebraic expressions in computer algebra. The important fact and formulae given below:

I. 'BODMAS' Rule: This rule depicts the correct sequence in which the operations are to be executed, so as to find out the value of a given expression.

Here 'B' stands for 'Bracket', 'O' stands for 'of', 'D' stands for 'Division', 'M' stands for 'Multiplication', 'A' stands for 'Addition', and 'S' stands for 'Subtraction'.

Thus, in simplifying an expression, first of all the brackets must be removed, strictly in the order:

(*i*) Of (*ii*) Division (*iii*) Multiplication

(*iv*) Addition (*v*) Subtraction

II. Modulus of a Real Number: Modulus of a real number a is defined as

$$|a| = \begin{cases} a, & \text{if } a > 0 \\ -a, & \text{if } a < 0 \end{cases}$$

Thus, $|5| = 5$ and $|-5| = -(-5) = 5$.

III. Virnaculum (or Bar): When an expression contains Virnaculum, before applying the 'BODMAS' rule, we simplify the expression under the virnaculum.

IV. Some Important Formulae:

(*i*) $(a + b)^2 = (a^2 + b^2 + 2ab)$

(*ii*) $(a - b)^2 = (a^2 + b^2 - 2ab)$

(*iii*) $(a + b)^2 + (a - b)^2 = 2(a^2 + b^2)$

(*iv*) $(a + b)^2 - (a - b)^2 = 4ab$

(*v*) $(a^2 - b)^2 = (a + b)\ (a - b)$

(*vi*) $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$

(*vii*) $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$

(*viii*) $(a^3 + b^3) = (a + b)\ (a^2 - ab + b^2)$

(*ix*) $(a^3 - b^3) = (a - b)\ (a^2 + ab + b^2)$

(*x*) $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$

(*xi*) $a^3 + b^3 + c^3 - 3abc = (a + b + c)\ (a^2 + b^2 + c^2 - ab - bc - ca)$

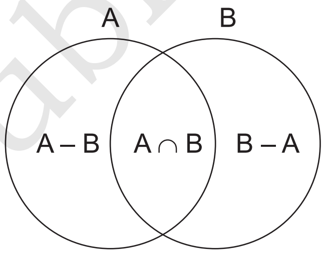
V. For any two sets A and B, we have:

(*i*) $n(\text{A} - \text{B}) + n(\text{A} \cap \text{B}) = n(\text{A})$

(*ii*) $n(\text{B} - \text{A}) + n(\text{A} \cap \text{B}) = n(\text{B})$

(*iii*) $n(\text{A} \cup \text{B}) = n(\text{A} - \text{B}) + n(\text{A} \cap \text{B}) + n(\text{B} - \text{A})$

(*iv*) $n(\text{A} \cup \text{B}) = n(\text{A}) + n(\text{B}) - n(\text{A} \cap \text{B})$



Example 1. *4368 + 2158 – 596 –? = 3421 + 1262*

Solution. Let $4368 + 2158 - 596 - x = 3421 + 1262$

$\Rightarrow \quad x + 596 = (4368 + 2158) - (3421 + 1262)$

$\Rightarrow \quad x + 596 = 6526 - 4683 = 1843$

$\Rightarrow \quad x = 1843 - 596 = 1247$

Hence, required number = 1247.

Example 2. *3456 ÷ 12 ÷ 8 = ?*

Solution. Given expression = $\frac{3456}{12} \div 8 = 288 \div 8 = 36.$

Example 3. *13 × 252 ÷ 42 + 170 = ? + 47*

Solution. Let $13 \times 252 \div 42 + 170 = x + 47$. Then

$$13 \times \frac{252}{42} + 170 = x + 47$$

$\Rightarrow \quad 13 \times 6 + 170 = x + 47$

$\Rightarrow \quad x + 47 = 78 + 170 = 248$

$\Rightarrow \quad x = 248 - 47 = 201.$

Hence required number = 201.

Factorization: Factorization can be defined as the process of breaking down a number into a smaller numbers which when multiplied together arrive at the original number. These numbers are broken down into factors or divisors. For example, 15 can be broken down as 3 × 5 and these two numbers are called factors.

Example 4. *Reduce the algebraic fractions to their lowest terms:*

$$\frac{x^2 - y^2}{x^3 - x^2y}.$$

Solution. $\frac{x^2 - y^2}{x^3 - x^2y}$

Factorizing the numerator and denominator separately and cancelling the common factors, we get

$$= \frac{(x + y)(x - y)}{x^2(x - y)} = \frac{x + y}{x^2}$$

Example 5. *Simplify the algebraic fractions* $\frac{36x^2 - 4}{9x^2 + 6x + 1}$

Solution. $\frac{36x^2 - 4}{9x^2 + 6x + 1}$

Step 1: Factorize the numerator: $36x^2 - 4$

$$= 4(9x^2 - 1) = 4[(3x^2) - (1)^2]$$
$$= 4(3x + 1)(3x - 1)$$

Step 2: Factorize the denominator: $9x^2 + 6x + 1$

$$= 9x^2 + 3x + 3x + 1$$
$$= 3x(3x + 1) + 1(3x + 1)$$
$$= (3x + 1) + (3x + 1)$$

Step 3: Simplification of the given expression after factorizing the numerator and the denominator:

$$= \frac{36x^2 - 4}{9x^2 + 6x + 1} = \frac{4(3x + 1)(3x - 1)}{(3x + 1)(3x + 1)}$$

$$= \frac{4(3x - 1)}{(3x + 1)}$$

15.2. METHOD OF SUBSTITUTION

In this method, we express one of the variables in terms of the other from one of the two equations and then substituting the value of the variable to the other equation to get an equation in one variable.

Steps:

1. Choose any one of the two given equations and find the value of one variable (say x) in terms of the other (say y)

   For example:

   $$a_1x + b_1y + c_1 = 0 \qquad \ldots(1)$$

   $$\Rightarrow \quad a_1x = -c_1 - b_1y$$

   $$\therefore \quad x = \frac{-c_1 - b_1y}{a_1}$$

2. Substitute the value of x, obtained in step 1 to the second equation to get an equation in y

   Example:

   $$a_2x + b_2y + c_2 = 0 \qquad \ldots(2)$$

   $$a_2 \times \frac{(-c_1 - b_1y)}{a_1} + b_2y + c_2 = 0$$

   $\therefore$ Equation (*ii*) reduces to an equation in y.

3. Solve the equation obtained in step 2 to get the value of y.
4. Substitute the value of y in equation (1) to get the value of x.
5. The values of x and y obtained in steps 3 and 4 are solution of pair of linear equations in two variables.

Example 6. *Solve the following pair of linear equations by substitution method:*

$$x + 2y = -1$$
$$2x + 3y = 12$$

Solution. (*i*) We have,

$$x + 2y = -1 \qquad \ldots(i)$$
$$2x + 3y = 12 \qquad \ldots(ii)$$

From (*i*), $\quad x = -1, -2y \qquad \ldots(iii)$

Substituting $x = -1 - 2y$ in (*ii*), we get

$$2(-1 - 2y) + 3y = 12$$

$$\Rightarrow \quad -2 - 4y + 3y = 12$$

$$\Rightarrow \quad -y = 12 + 2 = 14$$

$$\therefore \quad y = -14$$

Substituting $y = -14$ in (*iii*), we get

$$x = -1 - 2x - 14$$

$$\therefore \quad x = 27$$

So, $x = 27$ and $y = -14$ Ans.

Example 7. *Solve the following pair of linear equation by the substitution method.*

$$\frac{x}{3} + \frac{y}{4} = 11$$

$$\frac{5x}{6} - \frac{y}{3} = -7$$

Solution.

$$\frac{x}{3} + \frac{y}{4} = 11 \quad \ldots(i)$$

$$\frac{5x}{6} - \frac{y}{3} = -7 \quad \ldots(ii)$$

$$\Rightarrow \quad \frac{x}{3} + \frac{y}{4} = 11$$

$$\Rightarrow \quad \frac{x}{3} = 11 - \frac{y}{4}$$

$$\Rightarrow \quad \frac{x}{3} = \frac{44 - y}{4}$$

$$\Rightarrow \quad x = \frac{3(44 - y)}{4} \quad \ldots(iii)$$

Substituting $x = \dfrac{3(44 - y)}{4}$ in (*ii*), we get

$$\Rightarrow \quad \frac{5 \times 3(44 - y)}{6 \times 4} - \frac{y}{3} = -7$$

$$\Rightarrow \quad \frac{5}{8}(44 - y) - \frac{y}{3} = -7$$

$$\Rightarrow \quad \frac{15(44 - y) - 8y}{24} = -7$$

$\Rightarrow \quad 660 - 15y - 8y = -168$

$\Rightarrow \quad -23y = -168 - 660$

$\Rightarrow \quad 23y = 828$

$\Rightarrow \quad y = \dfrac{828}{23}$

$\therefore \quad y = 36$

Substituting $y = 36$ in (*iii*), we get

$$x = \frac{3(44-36)}{4}$$

$$\therefore \quad x = \frac{3 \times 8}{4} = 6$$

Hence solution of pair of equations is $x = 6$ and $y = 36$. **Ans.**

INEQUATION: A *statement involving variable(s) and the sign of inequality viz, >, <, ≥ or ≤ is called an inequation or an inequality.*

An inequation may contain one or more variables. Also, it may be linear or quadratic or cubic etc.

Following are some examples of inequations:

(*i*) $3x - 2 < 0$	(*ii*) $2x + 3 \le 0$
(*iii*) $5x - 3 > 0$	(*iv*) $4x + 5 \ge 0$
(*v*) $2x + 3y < 1$	(*vi*) $5x + 4y \le 3$
(*vii*) $4x - 6y > 5$	(*viii*) $2x + 5y \ge 4$
(*ix*) $2x^2 + 3x + 4 > 0$	(*x*) $x^2 - 3x + 2 \ge 0$
(*xi*) $x^2 + 3x + 2 < 0$	(*xii*) $x^2 - 5x + 4 \le 0$
(*xiii*) $x^3 - 6x^2 + 11x - 6 > 0$	(*xiv*) $x^3 + 6x^2 + 11x + 6 \le 0$

Linear Inequation in One Variable

Let a be a non-zero real number and x be a variable. Then inequations of the form ax + b < 0, ax + b ≤ 0, ax + b > 0 and ax + b ≥ 0 are known as linear inequations in one variable x.

For example, $9x - 15 > 0$, $5x - 4 \ge 0$, $3x + 2 < 0$ and $2x - 3 \le 0$ are linear inequations in one variable.

Linear Inequations in Two Variables

Let a, b be non-zero real numbers and x, y be variables. Then inequations of the form ax + by < c, ax + by ≤ c, ax + by > c and ax + by ≥ c are known as linear inequations in two variables x and y.

For example, $2x + 3y \leq 6$, $3x - 2y \geq 12$, $x + y < 4$, $2x + y \geq 6$ are linear inequations in two variables x and y.

Quadratic Inequation

Let a be A non-zero real number. Then an inequation of the form $ax^2 + bx + c < 0$, *or* $ax^2 + bx + c \leq 0$ *or* $ax^2 + bx + c > 0$ *or* $ax^2 + bx + c \geq 0$ *is known as a quadratic inequation.*

For example, $x^2 + x - 6 < 0$, $x^2 - 3x + 2 \geq 0$, $2x^2 + 3x + 1 > 0$ and $x^2 - 5x + 4 \leq 0$ are quadratic inequations.

15.3. SOLUTIONS OF AN INEQUATION

Definition: *A solution of an inequation is the value(s) of the variable(s) that makes it a true statements.*

Consider the inequation $\dfrac{3-2x}{5} < \dfrac{x}{3} - 4$.

Left hand side (LHS) of this inequation is $\dfrac{3-2x}{5}$ and right hand side (RHS) is $\dfrac{x}{3} - 4$

We observe that:

For $x = 0$, we have

$$\text{LHS} = \frac{3-2\times 9}{5} = -3 \text{ and RHS} = \frac{9}{3} - 4 = -1$$

Clearly, $-3 < -1$

$\Rightarrow$ LHS < RHS, which is true

So, $x = 9$ is a solution of the given inequation.

For $x = 6$, we have

$$\text{LHS} = \frac{3-2\times 6}{5} = -\frac{9}{5} \text{ and RHS} = \frac{6}{3} - 4 = -2$$

Because, $-\dfrac{9}{5} < -2$ is not true. So, $x = 6$ is not a solution of the given inequation.

We can verify that any real number greater than 7 is a solution of the given inequation.

Let us now consider the inequation $x^2 + 1 < 0$.

We know that

$$x^2 \geq 0 \text{ for all } x \in R$$

$$\therefore \quad x^2 + 1 \geq 1 \text{ for all } x \in R$$

$$\Rightarrow \quad x^2 + 1 \nless 0 \text{ for any } x \in R$$

So, there is no real value of x which makes the given inequation a true statement. Hence, it has no solution.

It follows from the above discussion that an inequation may or may not have a solution. However, if an inequation has a solution it may have infinitely many solutions.

Solving an Inequation: *It is the process of obtaining all possible solutions of an inequation.*

Solution Set: *The set of all possible solutions of an inequation is known as its solution set.*

For example, the solution set of the inequation $x^2 + 1 \geq 0$ is the set R of all real numbesr where as the solution set of the inequation $x^2 + 1 < 0$ is the null set ϕ.

15.4. SOLVING LINEAR INEQUATIONS IN ONE VARIABLE

As mentioned in the previous section that solving an inequation is the process of obtaining its all possible solutions. In the process of the solving an inequation, we use mathematical simplifications which are governed by the following rules:

Rule 1: *Same number may be added to (or subtracted from) both sides of an inequation without changing the sign of inequality.*

Rule 2: *Both sides of an inequation can be multiplied (or divided) by the same positive real number without changing the sign of inequality. However, the sign of inequality is reversed when both sides of an inequation are multiplied or divided by a negative number.*

Rule 3: *Any term of an inequation may be taken to the other side with its sign changed without affecting the sign of inequality.*

A linear inequation in one variable is of the form.

$ax + b < 0$ or $ax + b \leq 0$ or, $ax + b > 0$ or, $ax + b \geq 0$

Example 8. *Solve the following linear inequations:*

(*i*) $2x - 4 \leq 0$ (*ii*) $-3x + 12 < 0$

(*iii*) $4x - 12 \geq 0$ (*iv*) $7x + 9 > 30$

Solution. (*i*) We have,

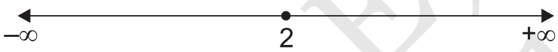
$$2x - 4 \leq 0$$

$$\Rightarrow \quad (2x - 4) + 4 \leq 0 + 4 \qquad \text{[Adding 4 on both sides]}$$

$$\Rightarrow \quad 2x \leq 4 \quad \Rightarrow \quad \frac{2x}{2} \leq \frac{4}{2} \quad \Rightarrow \quad x \leq 2$$

Hence, any real number less than or equal to 2 is a solution of the given inequation.

These solutions can be graphed on real line as shown in Figure.



The solution set of the given inequation is (– ∞. 2)

(*ii*) We have,

$$-3x + 12 < 0$$

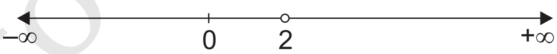
$$\Rightarrow \quad -3x < -12 \qquad \text{[Transposing 12 on right side]}$$

$$\Rightarrow \quad \frac{-3x}{-3} < \frac{-12}{-3} \qquad \text{[Dividing both sides by – 3]}$$

$$\Rightarrow \quad x > 4$$

Thus, any real number greater than 4 is a solution of the given inequation,

Hence, the solution set of the given inequation is (4, ∞). This solution set can be graphed on real line as shown in Figure.



(*iii*) We have

$$4x - 12 \geq 0$$

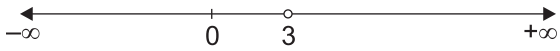
$$\Rightarrow \quad 4x \geq 12 \qquad \text{[Transposing 12 on RHS]}$$

$$\Rightarrow \quad \frac{4x}{4} \geq \frac{12}{4} \qquad \text{[Dividing both sides by 3]}$$

$$\Rightarrow \quad x \geq 3$$

$$\Rightarrow \quad x \in (3, \infty)$$

Hence, the solution set of the given inequation is (3, ∞). This solution set can be graphed on real line as shown in Figure.



(*iv*) We have

$$7x + 9 > 30$$

$\Rightarrow$ $7x > 30 - 9$ [Transposing 9 on RHS]

$\Rightarrow$ $7x > 21$

$\Rightarrow$ $\frac{7x}{7} > \frac{21}{7}$

$\Rightarrow$ $x > 3$

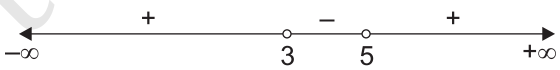
$\Rightarrow$ $x \in (3, \infty)$

Hence, (3, ∞) is the solution set of the given inequation. This can be graphed on real line as shown in Figure

Example 9. *Solve the following linear inequations:* $\frac{x-3}{x-5} > 0$

Solution. We have, $\frac{x-3}{x-5} > 0$

Equating $x - 3$ and $x - 5$ to zero, we obtain $x = 3, 5$ as critical points. Plot these points on real line as shown in Figure. The real line is divided into three regions. In the right must region the expression on LHS of equation is positive and in the remaining two regions it is alternatively negative and positive as shown in Figure.



Since the expression in (*i*) is positive, so the solution set of the given inequation is the union of regions containing positive signs. Hence from Figure.

$$\frac{x-3}{x-5} > 0 \Rightarrow x \in (-\infty, 3) \cup (5, \infty)$$

Hence, the solution set of the given inequation is $(-\infty, 3) \cap (5, \infty)$ as shown in Figure.

15.5. SOLUTION OF SIMULTANEOUS LINEAR INEQUATIONS IN TWO VARIABLE

In this section, we will discuss the technique of finding the solution set of simultaneous linear equations. Solving simultaneous linear inequations means finding the set of points (x, y) in which all the constraints are satisfied. Note that the solution set of simultaneous linear inequations may be an empty set or it may be the region bounded by the straight lines corresponding to linear inequations or it may be an unbounded region with straight lines boundaries.

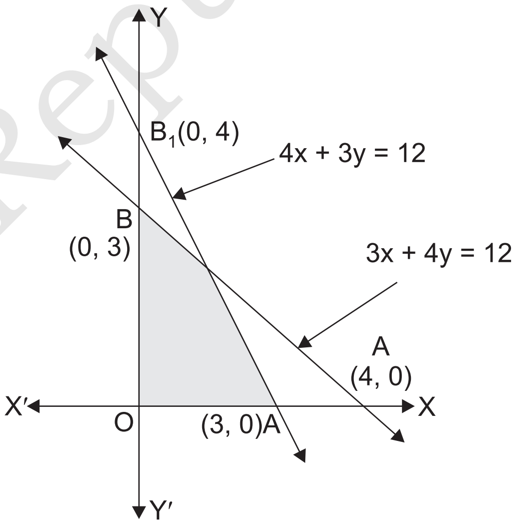
Example 10. *Exhibit graphically the solution set of the linear inequations.*

$3x + 4y \leq 12,\ 4x + 3y \leq 12,\ x \geq 0,\ y \geq 0$

Solution. Converting the inequations into equations, the inequations reduce to

$$3x + 4y = 12,\ 4x + 3y = 12,\ x = 0,\ y = 0$$

Region Represented by $3x + 4y \leq 12$: The line $3x + 4y = 12$ meets the coordinate axes at A(4, 0) and B(0, 3). Draw a thick line joining A and B. We find that (0, 0) satisfies inequation $3x + 4y \leq 12$. So, the portion containing the origin represents the solution set of the inequation $3x + 4y \leq 12$.



Region Represented by $4x + 3y \leq 12$: The line $4x + 3y = 12$ meets the x and y-axes at $A_1(3, 0)$ and $B_1(0, 4)$ respectively. Join these two points by a thick line. Clearly, the region containing the origin is represented by the inequation $4x + 3y \leq 12$.

Region Represented by $x \geq 0$ and $y \geq 0$: Clearly, $x \geq 0$ and $y \geq 0$ represent the first quadrant.

Hence, the shaded region given in Figure represents the solution set of the given linear inequations.

Example 11. *Exhibit graphically the solution set of the linear inequations.*

$$x + y \leq 5,\ 4x + y \geq 4,\ x + 5y \geq 5,\ x \leq 4,\ y \leq 3,$$

Solution. Converting the inequations into equations, we obtain

$$x + y = 5,\ 4x + y = 4,\ x + 5y = 5,\ x = 4,\ y = 3,$$

Region Represented by $x + y \leq 5$: The line $x + y = 5$ meets the coordinate axes at A(5, 0) and B(0, 5) respectively. Join these points by a thick line. Clearly (0, 0) satisfies the inequality $x + y \leq 5$. So, the portion containing the origin represents the solution set of the inequation $x + y \leq 5$.

Region Represented by $4x + y \geq 4$: The line $4x + y = 4$ meets the coordinate axes at $A_1(1, 0)$ and $B_1(0, 4)$ respectively. Join these points by a thick line. Clearly, (0, 0) does not satisfy the inequation $4x + y \geq 4$. So, the portion not containing the origin is represented by the inequation $4x + y \geq 4$.

Region Represented by $x + 5y \geq 5$: The line $x + 5y \geq 5$ meets the coordinate axes at A(5, 0) and $B_2(0, 1)$ respectively. Join these two points by a thick line. We find that (0, 0) does not satisfy the inequation $x + 5y \geq 5$. So, the portion not containing the origin is represented by the given inequation.

Region Represented by $x \leq 4$: Clearly, $x = 4$ is a parallel to y-axis at a distance of 4 units from the origin. Since (0, 0) satisfies the inequation $x \leq 4$. So, the portion lying on the left side of $x = 4$ is the region represented by $x \leq 4$.

Region Represented by $y \leq 3$: Clearly, $y = 3$ is a line parallel to y-axis at a distance 3 from it. Since (0, 0) satisfies $y \leq 3$. So, the portion containing the origin is represented by the given inequation.

The common region of the above five regions represents the solution set of the given linear constants as shown in Figure.

Simultaneous Linear Equations: A set of two or more equations each containing two or more variables whose values can simultaneously satisfy both or all the equations in the set, the numbers of variables being equal to or less than the number of equations in the set.

15.6 SOLVING SIMULTANEOUS LINEAR EQUATIONS

(*i*) Graphical method

1. Obtain the pair of linear equations

$$a_1x + b_1y + c_1 = 0 \qquad \text{...(1)}$$

and $$a_2x + b_2y + c_2 = 0 \qquad \text{...(2)}$$

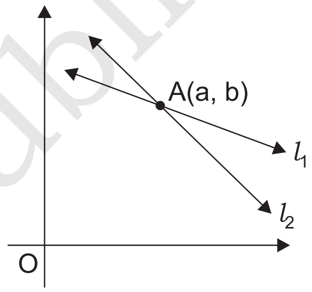
2. Find atleast two solution for each of the two equations by assuming value of one variable and then calculating the other variable.
3. Plot these points (solutions) of both the equation in the same co-ordinate axes to get two straight line, one for each equation.

While plotting the graph, the following three cases arises:

Case I: The two lines intersect at a point A (Figure)

Then the two equations have unique solutions given by $x = a$ and $y = b$.

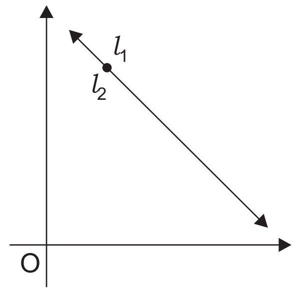
The equations are said to be consistent.



Case II: The two lines coincide each other (Figure)

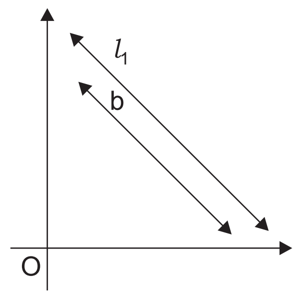
Then the two equations have infinitely many solutions.

The equations are said to be consistent.



Case III: The two lines are parallel to each other.

Then the two equations have no solutions and are said to be inconsistent.



Example 12. *Solve the following pair of equation graphically*

$$x + y = 3$$
$$2x + 5y = 12$$

Solution. We have,

$$x + y = 3 \qquad \ldots(1)$$

If $x = 1$, $y = 2$

If $x = 2$, $y = 1$

Two solutions are

x	1	2
y	2	1

and $2x + 5y = 12$

If $x = 1$, $2 \times 1 + 5y = 12 \Rightarrow 5y = 10$

$\therefore$ $y = 2$

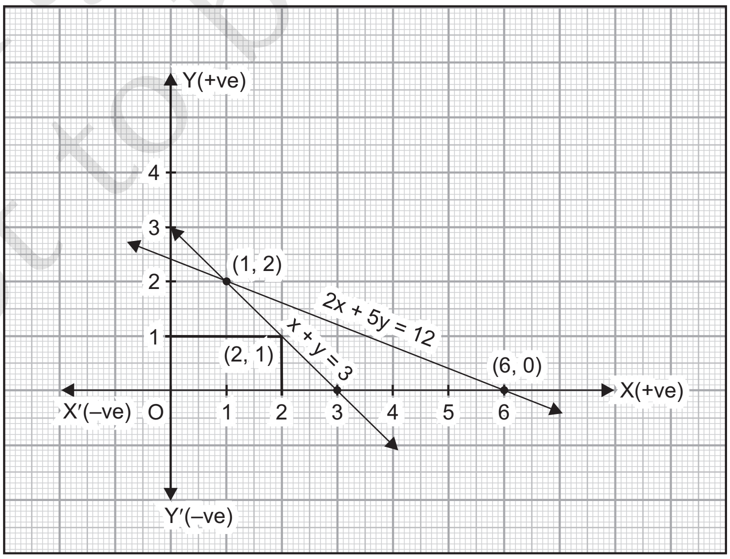
$x = 6$, $2 \times 6 + 5y = 12 \Rightarrow 5y = 0$

$\therefore$ $y = 0$

Two solutions are

x	1	6
y	2	0

Figure shows the graph of the equations. The two lines intersect at the point (1, 2).



15.7. QUADRATIC EQUATION

Any equation of the form $p(x) = 0$, where $p(x)$ is a polynomial of degree 2, is known as quadratic equation.

For Example: $2x^2 + 5x + 3 = 0$ is a quadratic equation.

An equation of the form $ax^2 + bx + c = 0$ where a, b, c are real number and $a \neq 0$ is known as the standard form of a quadratic equation.

15.8. ROOTS OF A QUADRATIC EQUATION

A real number α is a root of quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ if $a\alpha^2 + b\alpha + c = 0$. So any real number which satisfies a given quadratic equation is called the root of the quadratic equation.

In polynomial chapter, we learnt that if $x = \alpha$ satisfies a quadratic polynomial $p(x)$ *i.e.,* $p(\alpha) = 0$, then α is the zero of $p(x)$.

This means that zeroes of a quadratic polynomial $ax^2 + bx + c$ and roots of a quadratic equation $ax^2 + bx + c = 0$ are the same. A quadratic equation can have atmost two real roots.

15.9 SOLUTIONS OF A QUADRATIC EQUATION

Solving a quadratic equations means finding the roots of the quadratic equation.

Example 13. *Which of the following are quadratic equations?*

(*i*) $x^2 + 5x - 6 = 0$ (*ii*) $2x^2 + 6x + 8 = 0$

(*iii*) $x^2 - 5 = 0$ (*iv*) $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$

Solution. (*i*) As $x^2 + 5x - 6$ is a quadratic polynomial

$\therefore$ $x^2 + 5x - 6 = 0$ is a quadratic equation

(*ii*) $(2x^2 + 6x + 8)$ is a quadratic polynomial

$\therefore$ $2x^2 + 6x + 8 = 0$ is a quadratic equation.

(*iii*) $(x^2 - 5)$ is a quadratic polynomial

$\therefore$ $x^2 - 5 = 0$ is a quadratic polynomial.

(*iv*) $\sqrt{2}x^2 + 7x + 5\sqrt{2}$ is a quadratic polynomial

$\therefore$ $\sqrt{2}x^2 + 7x + 5\sqrt{2} = 0$ is a quadratic equation.

Linear Programming

Linear Programming is a method of finding an optimal value (i.e., maximum or minimum value) of a linear function of several variables subject to the conditions that variables are non-negative and satisfy a set of linear equations or inequations.

The linear function, which has to be maximised or minimised, is called objective function.

The process of maximisation or minimisation is called optimisation.

The variables involved in linear programming are called decision variables.

The restrictions on the decision variables to be non-negative are called non-negativity restrictions.

The restrictions on the decision variables to satisfy linear equations are called constraints.

In this chapter, we shall restrict our study to Linear Programming Problem in two variables and up to three non-trivial constraints. Also, we shall use the abbreviation L.P.P for the term 'Linear Programming Problem

A general L.P.P in two variables and up to three constraints is of the form

Maximise (or Minimise) $Z = ax + by\}$ Objective function

Subject to the constraints

$$\left.\begin{array}{l} a_1x + b_1y \{=, >, \geq, <, \leq\} c_1 \\ a_2x + b_2y \{=, >, \geq, <, \leq\} c_2 \\ a_3x + b_3y \{=, >, \geq, <, \leq\} c_3 \end{array}\right\} \text{Constraints}$$

$$x \geq 0,\ y \geq 0\} \quad \text{Non-negativity restrictions}$$

In Class XI, we have discussed the graphical method of solving system of linear inequalities in two variables. This method plays an important role in the study of Linear Programming Problems. Before proceeding further, let us recall this graphical method.

15.10. GRAPHICAL METHOD OF SOLVING L.P.P.

In this section, we shall discuss a graphical method of solving L.P.P. involving two decision variables x and y. This graphical method is known as **'Corner Point Method'**

We will first defined the following important terms used in this method.

1. **Feasible Region:** The common region determined by the constraints and non-negativity restrictions of L.P.P is called *feasible region.*
2. **Feasible Solution:** A set of values of decision variables of L.P.P. satisfying the constraints and the non-negativity restrictions is called *feasible solution.* Every point in the feasible region is a feasible solution of the given L.P.P.
3. **Optimal Feasible Solution/Optimal Solution:** A feasible solution of L.P.P is said to be *optimal feasible solution* (or *optimal solution*) if it optimises (*i.e.,* maximises or minimises) the objective function.
4. **Bounded Feasible Region:** A feasible region is said to be *bounded,* if it can be enclosed within a circle.
5. **Unbounded Feasible Region:** A feasible region is said to be *unbounded,* if it cannot be enclosed within any circle *i.e.,* if it extends indefinitely in any direction.
6. **Infeasible Region:** The region other than the feasible region is called *infeasible region.*
7. **Infeasible Solution:** Any point outside the feasible region is called *infeasible solution* of the given L.P.P.

The following theorems are fundamental in solving the L.P.P.

Theorem 1: Let R be the feasible region for L.P.P and let Z = $ax + by$ be the objective function. When Z has an optimum value (*i.e.,* maximum or minimum value), where the variables x and y are subject to the constraints described by the linear inequalities, this optimal value must occur at a corner point of the feasible region.

Theorem 2: Let R be the feasible region for L.P.P and let Z = $ax + by$ be the objective function. If R is bounded, then the objective function Z has both maximum or minimum value on R and each of these occurs at a corner point of R. If the feasible region is unbounded, then a maximum or a minimum may not exist. However, if it exists, it must occur at a corner point of R.

(The proofs of these theorems are beyond the scope of this book).

15.11. CORNER POINT METHOD

The Corner Point Method for solving L.P.P., involving two decision variables x and y, consists of the following steps:

Step I. Find the feasible region of the given L.P.P, and determine its corner points.

Step II. Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m be the largest and smallest values of Z at these corner points respectively.

Step III. If the feasible region is bounded, then M and m are maximum and minimum values of Z. If the feasible region is unbounded, then

(*i*) If the open half plane determined by $ax + by > M$ has no point in common with the feasible region, then M is maximum value of Z, otherwise Z has no maximum value.

(*ii*) If the open half plane determined by $ax + by < m$ has no point in common with the feasible region, then m is minimum value of Z, otherwise Z has no minimum value.

Remark: If Z has optimum (*i.e.,* maximum or minimum) value at any two corner points, then it has optimum value at all the points of the line segment joining those points.

Let us consider the following examples.

Example 14. *Maximise $Z = 3x + 4y$, subject to the constraints:*

$$x + y \leq 4, \quad x \geq 0, \quad y \geq 0$$

Solution. Given L.P.P is Maximise

$$Z = 3x + 4y$$

subject to the constraints

$$x + y \leq 4, \quad x \geq 0, \quad y \geq 0$$

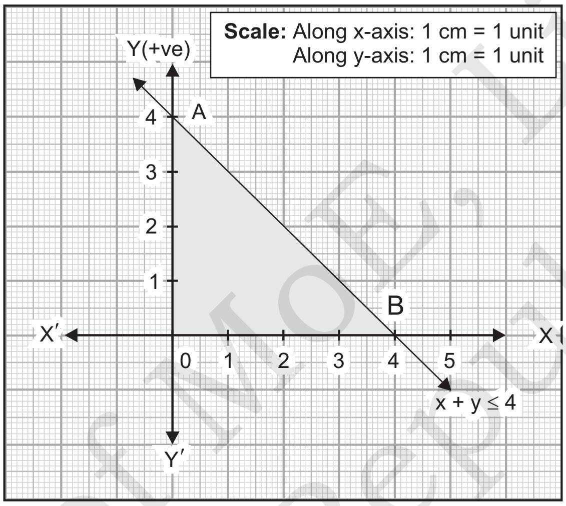
consider the following equations:

$x + y = 4$

x	0	4
y	4	0

$x = 0, \quad y = 0$

The feasible region of L.P.P. is bounded, as shown shaded in the graph.



Corner points	Value of Z(Z = 3x + 4y)
A(0, 4)	16
B(4, 0)	12
O(0, 0)	0

Since, the feasible region is bounded and 16 is the maximum value of Z at corner points.

Hence, 16 is the maximum value of Z in the feasible region at $x = 0$, $y = 4$.

Example 15. *Maximise $Z = 8x + 9y$ subject to the constraints:*

$$2x + 3y \leq 6, \quad 3x - 2y \geq 6, \quad y \leq 1, x \geq 0, y \geq 0.$$

Solution. Given L.P.P is

Maximise $Z = 8x + 9y$

Subject to the constraints:

$$2x + 3y \leq 6, \quad 3x - 2y \leq 6, \quad y \leq 1, x \geq 0, y \geq 0.$$

We consider the following equation

$2x + 3y = 6$

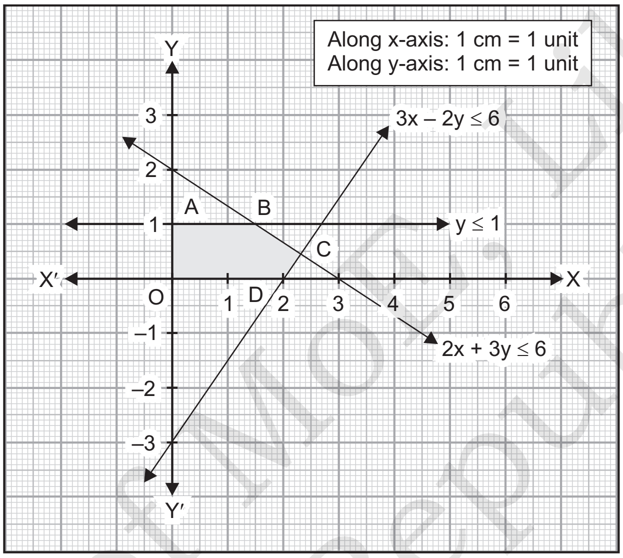
x	0	3
y	2	0

$3x - 2y = 6$

x	0	2
y	– 3	0

$y = 1,\ x = 0,\ y = 0$

The feasible region of L.P.P. is bounded, as shown shaded in the graph.



Corner points	Value of Z(Z = $8x + 9y$)
A(0, 1)	9
$B\left(\frac{3}{2}, 1\right)$	21
$C\left(\frac{30}{13}, \frac{6}{13}\right)$	$\frac{294}{13}$
D(2, 0)	16
O(0, 0)	0

Since, the feasible region is bounded and $\frac{294}{13}$ is the maximum value of Z at corner points.

Hence, $\frac{294}{13}$ is the maximum value of Z in the feasible region at $x = \frac{30}{13},\ y = \frac{6}{13}$.

15.12. WORD PROBLEMS ON LINEAR EQUATIONS

Worked-out word problems on linear equations with solutions explained step-by-step in different types of examples.

There are several problems which involve relations among known and unknown numbers and can be put in the form of equations. The equations are generally stated in words and it is for this reason we refer to these problems as word problems. With the help of equations in one variable, we have already practiced equations to solve some real life problems.

Steps involved in solving a linear equation word problem:

- Read the problem carefully and note what it is given and what is required and what is given.
- Denote the unknown by the variables x, y
- Translate the problem to the language of mathematics or mathematical statements.
- Form the linear equation in one variable using the conditions given in the problems.
- Solve the equation for the unknown.
- Verify to be sure whether the answer satisfies the conditions of the problem.

Example 16. *The length of a rectangle is twice its breadth. If the perimeter is 72 metre, find the length and breadth of the rectangle.*

Solution. Let the breadth of the rectangle be x,

Then the length of the rectangle = $2x$

Perimeter of the rectangle = 72

Therefore, according to the question

$$2(x + 2x) = 72 \quad \Rightarrow \quad 2 \times 3x = 72$$

$$\Rightarrow \quad 6x = 72 \quad \Rightarrow \quad x = 72/6$$

$$\Rightarrow \quad x = 12$$

We know, length of the rectangle = $2x$

$$= 2 \times 12 = 24$$

Therefore, length of the rectangle is 24 m and breadth of the rectangle is 12 m.

EXERCISE

1. (*a*) Simplify: (*a*) 460 × 15 – 5 × 20

(*b*) 1 ÷ (1 + 1 ÷ (1 + 1 ÷ 1(1 + 1 ÷ 2))) + 1

2. Find the missing numeral:

(*a*) (? – 2763) ÷ 86 × 13 = 208

(*b*) 3565 ÷ 23 + 4675 ÷ ? = 430

3. Simplify:

(*a*) $\frac{(6+6+6+6) \div 6}{4+4+4+4 \div 4}$ (*b*) $\frac{(2+3) \times 5 + 3 \div \frac{1}{2}}{6 + 5 \times 4 \div \frac{4}{5}}$

4. What should come in place of both the question marks in the following equation?

$$\frac{128 \div 16 \times ? - 7 \times 2}{7^2 - 8 \times 6 + ?^2} = 1$$

5. Factorise $4x^2 + 12x + 5$.

6. Factorise $y^2 + 16y + 60$.

7. Factorise $5x^2 + 14x - 3$.